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RESONANCES IN LINEAR ONE-DEGREE-OF-FREEDOM SYSTEMS WITH PIECEWISE-CONSTANT PARAMETERS†

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A technique based on the composition of elementary phase fluxes‡ is proposed for investigating parametric resonance in systems with "large" perturbations, described by second-order linear differential equations with periodic piecewise-constant coefficients. A monodromy matrix is given and a parametric resonance criterion is indicated, which takes into account the possibility of multiple multipliers and the action of dissipative forces. When there is a two-stage dependence of the coefficients on time during one period, regions of parametric resonance are obtained for different types of linear mechanical systems with one degree of freedom.

The technique of investigating parametric resonance has traditionally been based on a combination of Floquet's theory and small-parameter methods and Fourier analysis [1–4]. Another approach to this problem consists of using the conditions of absolute stability and variational methods as, for example, in [5]. To illustrate the results of the small-parameter method, an analogue of Hill's equation with piecewise-constant coefficients of the quasi-rectangular sine type was considered in [3], and the Meissner equation (a two-stage piecewise-constant analogue of the Mathieu equation) was considered in [1]. Meissner's equation was investigated in more detail in [2, 6]. This paper supplements and, in some cases, refines results obtained previously.

1. THE PARAMETRIC RESONANCE CRITERION

Suppose the motion of a mechanical system with one degree of freedom is described by a second-order ordinary linear differential equation, the coefficients of which are known periodic functions of time (or other independent variable) with the same period τ

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0 \tag{1.1}$$

We will denote by

$$A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$$

the monodromy matrix [4, 7], which gives the transformation of the solutions in terms of the period τ. It follows from Floquet's theorem [4] that the characteristic monodroy equation, which defines the multipliers, has the form

$$\mu^2 - B\mu + D = 0, \quad D = \exp\left(-\int_0^\tau a(t)dt\right)$$

where $B = a_{11} + a_{22}$ is the trace of the monodromy matrix.

By Floquet's theory the following criterion holds.‡

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Theorem 1. Suppose the motion of the system is described by a second-order homogeneous linear differential equation with periodic coefficients, having the same period. Parametric resonance occurs in it if and only if some of the following cases arise

1. $D > 1$,
2. $D = 1$: $|B| > 2$ or $|B| = 2$ and $a_{12}^2 + a_{21}^2 \neq 0$;
3. $0 < D < 1$: $|B| > D + 1$,

where a_{ij} are the components of the monodromy matrix.

This criterion differs from the one derived previously in [5] in that the necessary property of absolute stability of motion is not required.

2. CALCULATION OF THE MONODROMY MATRIX

In order to be able to use Theorem 1, we need to obtain the components of the monodromy matrix. Suppose the basis functions $x_1(t)$ and $x_2(t)$ satisfy the initial conditions [7]

$$x_1(0) = 1, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 1 \tag{2.1}$$

Then

$$a_{11} = x_1(\tau), \quad a_{22} = \dot{x}_2(\tau), \quad a_{21} = \dot{x}_1(\tau), \quad a_{12} = x_2(\tau)$$

We will henceforth investigate Eq. (1.1) with piecewise-constant coefficients

$$a(t) = \begin{cases} a_1, & 0 \leq t < t_1' \\ a_2, & t_1' \leq t < t_2' \\ \dots \\ a_n, & t_{n-1}' \leq t < t_n' = \tau \end{cases}, \quad b(t) = \begin{cases} b_1, & 0 \leq t < t_1' \\ b_2, & t_1' \leq t < t_2' \\ \dots \\ b_n, & t_{n-1}' \leq t < t_n' = \tau \end{cases}$$

where $a_k, b_k (t_k' = t_{k-1}' + t_k, t_k > 0; k = 1, \dots, n)$ are constant parameters and $t_0' = 0$.

Note that the intervals in which the values of the functions $a(t)$ and $b(t)$ are constant are identical. Consequently, in each such interval the differential equation defines the phase flux with matrix G_j, which transfers the phase point from the initial position at the instant of time t_{j-1}' to the final position at the instant of time t_j'.

We will use the rule of continuous extension of the solutions at instants when the coefficients jump. Suppose $A(t_j')$ is the matricant [4], corresponding to the instant of time $t_j' = t_{j-1}' + t_j$ and $A(t_{j-1}')$ is the matricant of the same solutions corresponding to the instant of time t_{j-1}'. Then

$$A(t_j') = G_j A(t_{j-1}')$$

The monodromy matrix is the value of the matricant $A(t)$ at the instant of time $t_n' = \tau$, and hence it is expressed by the formula

$$A = A(t_n') = G_1 G_2 \dots G_n$$

We will first consider the case when $b_k(t) = \omega_k^2(t) > 0$ $(k = 1, \dots, n)$, where all the coefficients of the equation take real values. This type of differential equation is conventionally called an oscillator equation. In particular, when $a(t) \equiv 0$ we have Hill's equation with piecewise-constant coefficients, and for this the phase-flux matrix is expressed in terms of the time interval t_j by the formula

$$G_j = \begin{Vmatrix} \cos(\omega_j t_j) & \dfrac{1}{\omega_j}\sin(\omega_j t_j) \\ -\omega_j \sin(\omega_j t_j) & \cos(\omega_j t_j) \end{Vmatrix}$$

When $a_j \neq 0$ and $a_j^2 < 4\omega_j^2$ $(j = 1, \dots, n)$ we have

$$G_j = \exp\left(-\frac{a_j t_j}{2}\right) G'_j, \quad \Omega_j = \sqrt{4\omega_j^2 - a_j^2}$$

$$G'_j = \begin{Vmatrix} \cos(\Omega_j t_j) + \dfrac{a_j}{2\Omega_j}\sin(\Omega_j t_j) & \dfrac{1}{\Omega_j}\sin(\Omega_j t_j) \\ -\left(\dfrac{a_j^2}{4\Omega_j} + \Omega_j\right)\sin(\Omega_j t_j) & -\dfrac{a_j}{2\Omega_j}\sin(\Omega_j t_j) + \cos(\Omega_j t_j) \end{Vmatrix}$$

If it turns out that $a_j^2 = 4\omega_j^2$, we obtain

$$G'_j = \begin{Vmatrix} 1 + a_j t_j/2 & t_j \\ -a_j^2 t_j/4 & 1 - a_j t_j/2 \end{Vmatrix}$$

Finally, if $a_j^2 > 4\omega_j^2$ the phase flux matrix takes the form

$$G_j = \frac{G_j^- \exp(\alpha_j^- t_j) + G_j^+ \exp(\alpha_j^+ t_j)}{\alpha_j^+ - \alpha_j^-}$$

$$\alpha_j^{\pm} = \frac{-a_j \pm \sqrt{a_j^2 - 4\omega_j^2}}{2}, \quad G_j^{\pm} = \begin{Vmatrix} \mp\alpha_j^{\mp} & \pm 1 \\ \mp\alpha_j^- \alpha_j^+ & \pm\alpha_j^{\pm} \end{Vmatrix}$$

For the oscillator equation the corresponding monodromy matrix is obtained in the form of the product of these elementary matrices, which describe the conversion of the phase coordinates in separate intervals in which the coefficients a and b are constant.

We consider the generalized oscillator equation with coefficient $a(t) \equiv 0$, $b_k = -\overline{\omega}_k^2$ $(k = 1, \ldots, n)$, where the constants $\overline{\omega}_k(t)$ can take either real or pure imaginary non-zero values. For this equation the phase-flux matrix in the jth interval has the form

$$G_j = \begin{Vmatrix} \operatorname{ch}(\overline{\omega}_j t_j) & \dfrac{1}{\overline{\omega}_j}\operatorname{sh}(\overline{\omega}_j t_j) \\ \overline{\omega}_j \operatorname{sh}(\overline{\omega}_j t_j) & \operatorname{ch}(\overline{\omega}_j t_j) \end{Vmatrix}$$

If it turns out that the quantity $\overline{\omega}_j = i\omega_j$ is pure imaginary, this matrix is identical with the phase-flux matrix of a harmonic oscillator.

Theorem 2. Suppose the solutions $x_1(t)$ and $x_2(t)$ of the generalized oscillator equation with coefficients $a(t) \equiv 0$, $b_k = -\overline{\omega}_k^2$ $(k = 1, \ldots, n)$ satisfy the initial conditions (2.1). Then the values of these solutions and their derivatives at the instant of time t'_n are expressed by the formulae

$$x_1(t'_n) = \frac{1}{2^{n-1}}\sum_\gamma P_\gamma \operatorname{ch} S_\gamma, \quad \dot{x}_1(t'_n) = \frac{\overline{\omega}_n}{2^{n-1}}\sum_\gamma \gamma_n P_\gamma \operatorname{sh} S_\gamma$$

$$x_2(t'_n) = \frac{1}{\overline{\omega}_1 2^{n-1}}\sum_\gamma P_\gamma \operatorname{sh} S_\gamma, \quad \dot{x}_2(t'_n) = \frac{\overline{\omega}_n}{\overline{\omega}_1 2^{n-1}}\sum_\gamma \gamma_n P_\gamma \operatorname{ch} S_\gamma$$

$$P_\gamma = \prod_{r=1}^{n-1}\left(1 + \frac{\gamma_r \overline{\omega}_r}{\gamma_{r+1}\overline{\omega}_{r+1}}\right), \quad S_\gamma = \sum_{r=1}^{n} \gamma_r \overline{\omega}_r t_r$$

where a set $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ consists of the company γ_r, which take the values $\gamma_1 = 1$, $\gamma_r = \pm 1$ $(r = 2, \ldots, n)$, while the summation is carried out over all these sets.

For Hill's equation with piecewise-constant coefficient $a(t) \equiv 0$, $b_k = -\omega_k^2$ $(k = 1, \ldots, n)$ we can assume $\overline{\omega}_k = i\omega_k$. Then the components of the monodromy matrix are given by the following corollary.

Corollary 1. Suppose the solutions $x_1(t)$ and $x_2(t)$ of Hill's equation with piecewise-constant functions $\omega(t)$ satisfy initial conditions (2.1) when $t = 0$. Then the assertion of Theorem 2 holds when $\bar{\omega}_k$ is replaced by ω_k and the hyperbolic functions are replaced by corresponding trigonometric functions with a change in the sign of the expression for $\dot{x}_1(t'_n)$ obtained in this way.

3. TWO-STAGE PIECEWISE-CONSTANT ACTION

Using the formulae of the previous section we will investigate the resonance regions for several systems described by different types of differential equations.

System 1. Suppose the motion is described by Hill's equation with piecewise-constant periodic function $b = \omega^2(t)$, where

$$\omega(t) = \begin{cases} \omega_1, & 0 \leqslant t < t_1 \\ \omega_2, & t_1 \leqslant t < t_2' = t_1 + t_2 = \tau \end{cases} \tag{3.1}$$

We will use Theorem 1 and take into account the fact that $D = 1$. We will first investigate the resonance condition $|a_{11} + a_{22}| > 2$. We will put $\tau_1 = \omega_1 t_1$, $\tau_2 = \omega_2 t_2$. In accordance with Corollary 1, the condition considered takes the form

$$|\varkappa_+\eta_+ - \varkappa_-\eta_-| > 2 \tag{3.2}$$

where $\varkappa_\pm = (\omega_1 \pm \omega_2)^2/(2\omega_1\omega_2)$, $\eta_\pm = \cos(\tau_1 \pm \tau_2)$. Obviously for any positive values of ω_1, ω_2 we have $\varkappa_+ \geqslant 2$, $\varkappa_- \geqslant 0$. The equalities $\varkappa_+ = 2$, $\varkappa_- = 0$ are obtained if and only if $\omega_1 = \omega_2$ The parametric-resonance condition can be represented in the form of the set of inequalities

$$\eta_+ > \frac{2}{\varkappa_+} + \frac{\varkappa_-}{\varkappa_+}\eta_- \quad \text{or} \quad \eta_+ < -\frac{2}{\varkappa_+} + \frac{\varkappa_-}{\varkappa_+}\eta_-$$

In order to obtain the resonance region, we must cut a strip from the square $|\eta_-| \leqslant 1$, $|\eta_+| \leqslant 1$ between the straight lines

$$\eta_+ = \frac{2}{\varkappa_+} + \frac{\varkappa_-}{\varkappa_+}\eta_-, \quad \eta_+ = -\frac{2}{\varkappa_+} + \frac{\varkappa_-}{\varkappa_+}\eta_-$$

Note that $\varkappa_-/\varkappa_+ < 1$, so that the slope of these lines does not exceed $\pi/4$. The first of these will then pass through the point (1, 1) while the second will pass through the point (−1, −1).

In Fig. 1 the regions corresponding to parametric resonance are shown hatched. They exist for any ω_1 and ω_2 that are not equal to one another and only disappear when $\omega_1 = \omega_2$.

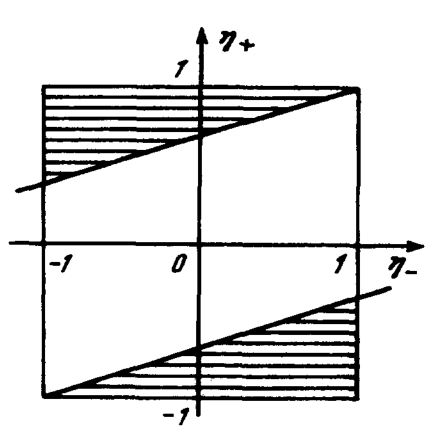

Fig. 1.

According to Theorem 1 there is no resonance in the case of multiple multipliers if and only if the following equalities are simultaneously satisfied

$$a_{21} = a_{12} = 0$$

Taking the notation employed into account we have a homogeneous system of linear equations in the unknowns $\sin(\tau_1 + \tau_2)$, $\sin(\tau_1 - \tau_2)$. If $\omega_1 \neq \omega_2$, the determinant of the system is non-zero, and its solution is

$$\sin(\tau_1 + \tau_2) = 0, \quad \sin(\tau_1 - \tau_2) = 0$$

which corresponds to the corners of the square in Fig. 1. Hence, parametric resonance will also occur in the case of multiple multipliers, and the hatched regions in Fig. 1 include the boundary, with the exception of the points (1, 1) and (–1, –1).

System 2. Suppose the motion in a small neighbourhood of an unstable equilibrium is described by an equation with coefficients $a \equiv 0$, $b = -\omega^2(t)$, where the function $\omega(t)$ has the form (3.1). We will assume that $D = 1$. We will investigate the resonance condition $|a_{11} + a_{22}| > 2$. In accordance with Theorem 2 the condition in question takes the form (3.2), where the coefficients $\varkappa_\pm$ are the same as in system 1 and $\eta_\pm = \operatorname{ch}(\tau_1 \pm \tau_2)$. In the plane of the variables (η_+, η_-) the region of permissible values is the corner defined by the inequalities $\eta_- \geqslant 1$; $\eta_+ \geqslant \eta_-$, since $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$. As might have been expected, this corner belongs to the region of parametric resonance, with the exception of the degeneracy point (1.1), for which $\tau_1 = \tau_2 = 0$.

System 3. Suppose $a \equiv 0$, $b = -\omega^2(t)$, where $\bar{\omega}$ can take both real and pure imaginary values, for example

$$\bar{\omega}(t) = \begin{cases} \omega_1, & 0 \leqslant t < t_1 \\ i\omega_2, & t_1 \leqslant t < t_2' = t_1 + t_2 = \tau \end{cases}$$

By Theorem 2 the condition $|a_{11} + a_{22}| > 2$ can be represented in the form

$$\eta > 1 + \varkappa\xi \quad \text{or} \quad \eta < -1 + \varkappa\xi, \quad \varkappa = \frac{1}{2}\left(\frac{\omega_1}{\omega_2} - \frac{\omega_2}{\omega_1}\right)$$

where $\eta = \operatorname{ch}\tau_1 \cos\tau_2$, $\xi = \operatorname{sh}\tau_1 \sin\tau_2$. In the (ξ, η) plane, in the region between the parallel straight lines $\eta = 1 + \varkappa\xi$ and $\eta = -1 + \varkappa\xi$, which always exist for finite values of $\varkappa$, parametric resonance does not occur. The curves corresponding to the change in the parameter τ_2 for a fixed value of the parameter τ_1, are closed ellipses. Consequently, for any ω_1 and ω_2 we can obtain a combination of the parameters τ_1 and τ_2 for which the motion in the neighbourhood of an unstable position of equilibrium will be stabilized.

System 4. We will investigate how a small dissipation, added to Hill's equation, can change the resonance region. The corresponding system is described by differential equation (1.1), in which $b = \omega^2$, $\omega(t)$ is expressed by (3.1) and

$$a(t) = \begin{cases} a_1, & 0 \leqslant t < t_1 \\ a_2, & t_1 \leqslant t < t_1 + t_2 = \tau \end{cases}$$

where $a_1 \geqslant 0, a_2 \geqslant 0$ are constants, and $\Omega_1^2 = 4\omega_1^2 - a_1^2 > 0$, $\Omega_2^2 = 4\omega_2^2 - a_2^2 > 0$. The monodromy matrix for this case is obtained in the form of the product of corresponding elementary matrices of the phase flux (see Section 2): $A = \sqrt{(D)}G_2'G_1'$.

We use Theorem 1 for the case $D < 1$. Multiplying the matrices, we obtain the parametric-resonance condition

$$|\varkappa_+\eta_+ - \varkappa_-\eta_-| > 2\operatorname{ch}\left(\frac{a_1t_1 + a_2t_2}{2}\right)$$

$$\varkappa_\pm = \frac{4(\Omega_1 \pm \Omega_2)^2 - (a_1 + a_2)^2}{8\Omega_1\Omega_2}, \quad \eta_\pm = \cos(\Omega_1t_1 \pm \Omega_2t_2)$$

We will assume that the coefficients a_1 and a_2 are so small that $\varkappa_\pm > 0$. It is then obvious that the maximum of the modulus of the left-hand side of the criterion is reached when $\eta_+\eta_- = -1$ and is equal to $\varkappa_+ + \varkappa_-$. A sufficient condition for parametric resonance to be possible in the system will be

$$\varkappa_+ + \varkappa_- > 2\,\mathrm{ch}\left(\frac{a_1t_1 + a_2t_2}{2}\right) \tag{3.3}$$

In particular, if $\eta_+ = 1$ and $\eta_- = -1$, we have

$$\Omega_1 t_1 = \frac{\pi}{2} + \pi(l+k), \quad \Omega_2 t_2 = -\frac{\pi}{2} + \pi(l-k), \quad l,\ k = 0,\ \pm 1,\ \pm 2, \ldots$$

Therefore

$$\frac{a_1t_1 + a_2t_2}{2} = \frac{1}{2}\left[\left(\frac{a_1}{\Omega_1} - \frac{a_2}{\Omega_2}\right)\left(\frac{\pi}{2} + \pi k\right) + \left(\frac{a_1}{\Omega_1} + \frac{a_2}{\Omega_2}\right)\pi l\right]$$

Depending on the dissipation, condition (3.3) thereby limits the range of values of the natural numbers l and k which specify the relation between the durations of the sections in which the coefficients of differential equation (1.1) are constant.

System 5. Suppose the equation of motion has the same for m as for System 4, but for each interval in which the coefficients are constant the condition $a_j^2 = 4\omega_j^2, j = 1; 2$ holds. The condition for parametric resonance is equivalent to the inequality

$$\frac{(a_1 - a_2)^2}{a_1a_2} > \frac{2[1 + \mathrm{ch}(\tau_1 + \tau_2)]}{\tau_1\tau_2}$$

where $\tau_1 = a_1t_1/2$, $\tau_2 = a_2t_2/2$. For fixed values of $\tau_1 > 0$, $\tau_2 > 0$ this inequality has a solution for which parametric resonance will occur, despite the dissipative nature of the forces acting on the system.

System 6. Suppose the dissipation is so large that $a_j^2 = 4\omega_j^2$. The trace of matrix A is expressed by the formula

$$\mathrm{tr}\,A = y_1y_2 + z_1z_2 - \varkappa(y_1 - z_1)(y_2 - z_2)$$

where

$$\varkappa = \frac{(\alpha_1^+ - \alpha_2^+)(\alpha_1^- - \alpha_2^-)}{(\alpha_2^+ - \alpha_2^-)(\alpha_1^+ - \alpha_1^-)}$$

$$0 < y_1 = \exp(\alpha_1^- t_1) < z_1 = \exp(\alpha_1^+ t_1) < 1$$

$$0 < y_2 = \exp(\alpha_2^- t_2) < z_2 = \exp(\alpha_2^+ t_2) < 1$$

If the times t_1 and t_2 are specified, we have

$$\alpha_1^- = \frac{\ln y_1}{t_1}, \quad \alpha_1^+ = \frac{\ln z_1}{t_1}, \quad \alpha_2^- = \frac{\ln y_2}{t_2}, \quad \alpha_2^+ = \frac{\ln z_2}{t_2}$$

$$\mathrm{tr}\,A = z_1z_2 + y_1y_2 - \frac{(t_2\ln z_1 - t_1\ln z_2)(t_2\ln y_1 - t_1\ln y_2)}{(\ln z_1 - \ln y_1)(\ln z_2 - \ln y_2)}(y_1 - z_1)(y_2 - z_2)$$

The quadratic form with respect to times t_1 and t_2 in the numerator of the fraction can be made as large in absolute value as desired and, consequently, the conditions for parametric resonance can always be satisfied in this case.

System 7. We will use the results of the analysis of System 1 and consider the resonance conditions for Hill's equation, in which

$$\omega(t)=\begin{cases}(1+\varepsilon)\omega, & 0\leqslant t<\tau/2\\ (1-\varepsilon)\omega, & (\tau/2)\leqslant t<\tau, \quad \varepsilon<1\end{cases}$$

where $\tau = 2\pi/\nu$. We have Meissner's equation [2]. This equation was considered in [1, 3] in order to illustrate the asymptotic estimates of the resonance regions for small values of ε. We will indicate the resonance regions without assuming that ε is small. A less complete result was obtained previously in [2, 6]. In the notation of System 1 we will have

$$\omega_1=(1+\varepsilon)\omega,\quad \omega_2=(1-\varepsilon)\omega,\quad t_1=t_2=\frac{\tau}{2}$$

$$\varkappa_+=\frac{2}{1-\varepsilon^2},\quad \varkappa_-=\frac{2\varepsilon^2}{1-\varepsilon^2},\quad \tau_1=\frac{\omega\tau(1+\varepsilon)}{2},\quad \tau_2=\frac{\omega\tau(1-\varepsilon)}{2}$$

Since $D = 1$, we obtain two cases of parametric resonance.

Case 1: $\varepsilon^2\sin^2(\chi\varepsilon) \geqslant \sin^2\chi$.

Case 2: $\varepsilon^2\cos^2(\chi\varepsilon) \geqslant \cos^2\chi$, $\chi = \omega\tau/2 = \pi\omega/\nu$.

In Fig. 2 we show the regions of the resonance relations of the parameters. The regions that are showed hatched from left to right downwards correspond to Case 1, while the regions that are shown hatched from left to right upwards correspond to Case 2. The quantities y_i, z_i ($i = 1, 2, \ldots$) are the successive roots of the equations $\pi y = \operatorname{ctg} \pi y$, $\pi z = -\operatorname{tg} \pi z$, respectively. These roots give the arguments of the maxima for the functions $f_1(\varepsilon) = \varepsilon^2\sin^2\chi\varepsilon$, $f_2(\varepsilon) = \varepsilon^2\cos^2\chi\varepsilon$ for a fixed ratio ω/ν. For example, the values of the argument ε for which the function $f_1(\varepsilon)$ reaches a maximum are expressed in the formula $\varepsilon_i = z_i\nu/\omega$. The arguments of the maxima for the function $f_2(\varepsilon)$ are calculated in exactly the same way. The dashed curves correspond to these arguments of the maxima. The zeros of the functions $f_1(\varepsilon)$ and $f_2(\varepsilon)$ are calculated in the same way. The graphs of the corresponding inversely proportional relations are represented by the continuous thin curves. At the boundary of the hatched regions, parametric resonance will also occur, with the exception of points for which $\sin 2\chi = 0$, $\sin 2\chi\varepsilon = 0$ simultaneously. The values $\omega/\nu = n/2$, $\varepsilon = k/n$ correspond to these points, where k and n are integers. The points in Fig. 2 which generate resonance regions when $\varepsilon = 0$, correspond to the points $\eta_+ = \pm 1$, $\eta_- = 0$ in Fig. 1. This supplements the well-known pattern of resonance regions in this case [6]. Note also that, unlike the results obtained in [2], the boundaries of the regions of parametric resonance do not contain self-intersections.

System 8. Consider the oscillations of a pendulum of variable length $l(t)$ in a parallel gravity field. We will write its equation of motion in the form

$$\ddot\varphi+\frac{2\dot l}{l}\dot\varphi+\frac{g}{l}\sin\varphi=0$$

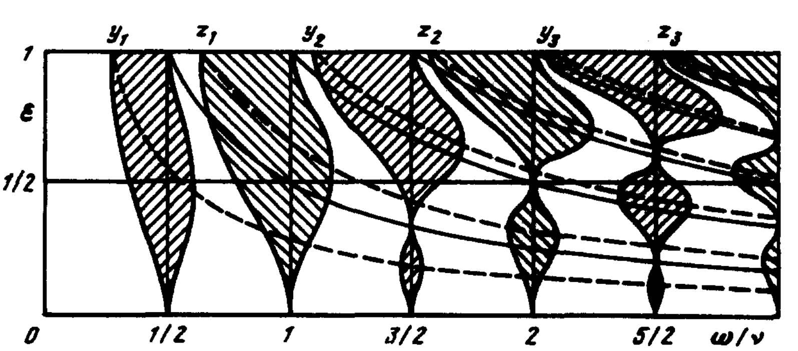

Fig. 2.

We have $a(t) = 2\dot{l}/l$, $D = 1$, irrespective of the form of the function $l(t)$. We will assume that the length of the pendulum varies periodically as given by

$$l(t) = \begin{cases} l_1, & 0 \leqslant t < t_1 \\ l_2, & t_1 \leqslant t < t_1 + t_2, \quad \tau = t_1 + t_2 \end{cases}$$

When t varies inside the limits in which the function $l(t)$ is constant, the equation of the oscillations of a mathematical pendulum will hold, which, for small oscillations, can be represented approximately in the form of Hill's equation with a function $\omega(t)$ of the form (3.1), where $\omega_1 = \sqrt{(g/l_1)}$, $\omega_2 = \sqrt{(g/l_2)}$.

All this recalls the formulation of the problem for System 1 considered above. The difference is that at the instants when there is an abrupt (very rapid) change in the length of the pendulum we cannot neglect the term containing the product of the velocities $\dot{l}\dot{\varphi}$ in its equation of motion. In Hill's equation, when there was a sudden change in the function ω the coordinate and velocity remained continuous. In the example considered, the coordinate φ will be continuous at the instant of reversal, while the velocity $\dot{\varphi}$, by the theorem on the kinetic moment about the point of suspension, changes abruptly together with a change in the length of the pendulum

$$l_2^2\dot{\varphi}^+ = l_1^2\dot{\varphi}^- \to \dot{\varphi}^+ = \lambda^2\dot{\varphi}^-, \quad \lambda = l_1/l_2$$

where $\dot{\varphi}^+$ is the value of the angular velocity after the sudden change in the length of the pendulum, while $\dot{\varphi}^-$ is the change in the angular velocity before the change. When the length changes from l_2 to l_1, the corresponding coefficient for recalculating the angular velocity changes its sign.

We can now use the parametric resonance criterion, taking $\tau_1 = \omega_1 t_1$, $\tau_1 = \omega_2 t_2$. We will write the components of the monodromy matrix as

$$a_{11} = \cos\tau_1\cos\tau_2 - \lambda^2\omega_1\omega_2^{-1}\sin\tau_1\sin\tau_2$$

$$a_{21} = -\lambda^{-2}\omega_2\cos\tau_1\sin\tau_2 - \omega_1\sin\tau_1\cos\tau_2$$

$$a_{12} = \omega_1^{-1}\sin\tau_1\cos\tau_2 + \lambda^2\omega_2^{-1}\cos\tau_1\sin\tau_2$$

$$a_{22} = -\lambda^{-2}\omega_1\omega_2\sin\tau_1\sin\tau_2 + \cos\tau_1\cos\tau_2$$

Note that in the formulae for a_{21} and a_{22} there is factor λ^{-2}, which is due to the fact that the period of the function $l(t)$ is completed when it takes the value which it had at the beginning of the period.

According to Theorem 1 the resonance condition $|a_{11} + a_{22}| > 2$ takes the form (3.2), where $\varkappa_\pm = (\omega_1\lambda \pm \omega_2\lambda^{-1})^2/(2\omega_1\omega_2)$. It is obvious that for any positive values of l_1 and l_2 we have $\varkappa_+ \geqslant 2$, $\varkappa_- \geqslant 0$. The equalities $\varkappa_+ = 2$, $\varkappa_- = 0$ occur if and only if $l_1 = l_2$.

In the case of multiple multipliers, the conclusions obtained for System 1 remain the same. Finally, we again have the resonance regions shown in Fig. 1.

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